Washington State University

Math 182

Honors Calculus II

# Euler's Identity

Concerning the real, the unreal, and the infinite

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# 1 Historical Background

Leonhard Euler was one of the most influential mathematicians to contribute to the theory and application of mathematics. One of his greatest achievements was discovering the simple yet substantial identity, $e^{ix} = \cos(x) + i\sin(x)$. Euler's formula is known for its combination of the five most important numbers in the mathematical world. Though Euler had enjoyed many other successes, it is appropriate to separately underscore this one, as it has many different forms and functions, as well as having a large-scale effect on the understanding of different areas of work and study.

Leonhard Euler, born in Basel, Switzerland, in 1707, was fortunate to begin his life with a well-rounded education, considering his father's wish that Euler take up ministry as his livelihood. Euler had the good fortune in his childhood to come into contact with perhaps the greatest mathematician at the time, Johann Bernoulli, who had been a friend of Euler's father during their undergraduate studies. This relationship was crucial to Euler's education, as Bernoulli encouraged an increase in mathematical studies, tutoring him when necessary. Bernoulli would suggest books for Euler to read, and then Euler was permitted to visit him once a week and ask any questions concerning the topics and theories that he did not fully comprehend. In this manner, Euler began to discover his abilities in mathematics, and at the same time, Bernoulli also began to see his great potential in the mathematical world. Euler originally entered into theology school with the intention of continuing on in the ministry, but with Bernoulli's conviction of Leonhard's potential in mathematics he switched his major and became fully engrossed in the world of mathematics (O'Connor [7]).

Through Bernoulli's detached training, along with the influence of substantial figures with whom he came into close contact while working at the St. Petersburg Academy of Sciences, Euler was able to learn and study new frontiers of mathematics. Monthly he would write an article concerning math and science in the journal published through the Academy, a practice that continued for nearly the rest of his life. Along with these articles, Euler published his own books, including subjects in cartography, science education, magnetism, fire engines, machines, and ship building, which directed him to his core areas of study, including number theory, infinite analysis, differential equations, calculus of variations, and rational mechanics (O'Connor [7]).

Some of Euler's greatest works were created when he moved back to Berlin for a 25 year period. These included many books on calculus of variations, calculation of planetary orbits, artillery and ballistics, analysis, shipbuilding and navigation, motion of the moon, and his most popular book, the *Letters to a Princess of Germany.* This latter book was very popular, translated into many languages and even exported to the Americas for people to read. The text involved popular science mechanics and was widely read (O'Connor [7]).

Though his most popular work may have dealt a great deal with the general mechanical sciences, the books of most interest and importance involved his vast knowledge and understanding of mathematical concepts such as complex numbers and infinite series. One such book is still known to this day as being a critical learning instrument for calculus students everywhere, as it is translated in French, German, Russian, and English. Introduction to Analysis of the Infinite covers a great deal concerning series and their deviations. In one such chapter of the book he discusses the relationship between the infinite series of the sine and cosine functions. From this comes the greatest and, by popular vote of mathematicians worldwide, the most beautiful mathematical identity known to man (Nahin [3], pg. 2): $e^{ix} = \cos(x) + i\sin(x)$. Euler's Identity was discovered in 1740 within letters sent to his old mentor Bernoulli. In these letters he was solving differential equations and discovered the famous identity (Nahin [3], pg. 143). This equation has had a significant impact on the world of mathematics and science, changing many different areas of work and study, and should be noted as one of his greatest and most significant contributions.

# 2 Mathematics of Euler's Identity

## 2.1 Derivation

Many theorems and identities that form a foundational link between various mathematical disciplines can be derived through diverse methods. Common equations such as the Pythagorean Theorem have as many as 371 different proofs involving topics ranging from simple geometric explanations to advanced calculus (Maor [1], pg. xiii). Although Euler's Identity has not been proved in such a large quantity of unique instances, it has manifested itself in a variety of forms and locations throughout the realm of mathematics. While Euler was solving the differential equation, $y'' + y = 0$, he found four solutions. The first two are rather obvious, $y = \sin(x)$ and $y = \cos(x)$. The others that Euler found were $y = e^{ix}$ and $y = e^{-ix}$. This strange mix of trigonometric and exponential equations shows that the sine and cosine functions must be some combination of $e$ and the imaginary number as shown. In the letter to Bernoulli, he explained the results of this differential equation and combined the four answers into a startling formula: $e^{ix} = \cos(x) + i\sin(x)$.

This amazing relation between the ordinary trigonometric functions, imaginary numbers, and $e$ was only the beginning. In his book, *Introduction to Analysis of the Infinite*, Euler formally produced his formula with an in-depth discussion of its properties, and, as the title suggests, explored the relation of his formula to that of infinite series. This exploration and its implications is printed and reprinted as a popular explanation or proof of Euler's formula.

To begin with, it should be noted that many continuous, differentiable functions can be written as the sum of a sequence of numbers (Stewart [2], pg. 462). Two of the most fundamental functions of trigonometry and calculus can be written as a series, called a Taylor series, sine and cosine.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{1}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{2}$$

The final piece to the puzzle comes again from the Taylor series for $e^x$, with $e$ being arguably one of the most important numbers ever discovered aside from $\pi$. This power series for $e^x$ was used by both Bernoulli and Euler in some spectacular calculations. Euler even used it to calculate the value of $e$ correct to 23 digits (Stewart [2], pg. 458)! Through a process of taking derivatives and solving for coefficients, $e^x$ can be shown to be:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3}$$

Euler's formula is derived by an ingenious substitution of $ix$ for $x$ in Equation 3.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = 1 + \frac{ix}{1!} + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \cdots$$

By grouping all the even powers and odd powers into their respective groups, and factoring the imaginary number from all of the odd powers, we see that:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Euler made a simple observation, viz., that these are indeed the functions for sine and cosine as shown above by the Taylor series representation, which resulted in his amazing formula:

$$e^{ix} = \cos(x) + i\sin(x) \tag{4}$$

## 2.2 Theoretical Uses of Euler's Identity

Although Euler's Identity may be of little interest to the less informed, its impact on mathematics has been a ground shaking event that gives mathematicians and scientists the tools to dip into the complex arena. In a sense, this formula is divulging the secrets of complex space, and gives us a window through which we can peer into the "other side." What follows will give a small taste of this marked penetration obtained by the formula.

### 2.2.1 Logarithms of Negative Numbers

Trigonometry students have always been taught that they cannot take logarithms of negative numbers—this would not yield a logical outcome. Yet, with the advent of Euler's formula, this bizarre idea becomes a reality. Using the Identity and substituting $\pi$ in for $x$ gives the famous version of the formula, $e^{i\pi} = -1$. The sine function was simply reduced to zero and the cosine took on the value of $-1$. Now it is trivial to take the natural logarithm of both sides, giving us a formula for the natural logarithm of negative one $\ln(-1) = i\pi$. In truth, this is misleading because multiple values of $x$ could produce the same effect of removing the complex term in Euler's formula. In fact, any odd multiple of $\pi$ will give an expression for $\ln(-1)$:

$$\ln(-1) = i(2n-1)\pi.$$

Further manipulation and raising to powers can give the natural logarithms of other negative numbers. Euler's Identity has given greater understanding of the nature of complex numbers as never before.

### 2.2.2 Raising Complex numbers to powers of Complex Numbers

The sheer depth of Euler's formula and the fact that it somehow ties the real and complex number systems together through a simple relation gives rise to the ability to compute complex powers. By simply substituting $x = \frac{\pi}{2}$ into the original equation, Euler's formula reduces to $e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$. Now by simply raising both sides to the power of $i$, it turns out that:

$$i^i = e^{\frac{-\pi}{2}} \approx .2078796.$$

This is equivalent to saying, $\sqrt{-1}^{\sqrt{-1}} = e^{\frac{-\pi}{2}}$. It is surprising to note that $i$ raised to itself is indeed a real number. Concepts such as this have profound implications for complex number theory by allowing us to manipulate complex numbers even though they are difficult to comprehend. It is said that the American mathematician Benjamin Peirce ($1809-1880$) proved this formula, $i^i = e^{\frac{-\pi}{2}}$, for his Harvard class and dramatically announced to the students: "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it is the truth." (Nahin [4], pg. 68) Euler's formula provides the tools needed to expand the knowledge of the operation of complex numbers.

### 2.2.3 One raised to irrational powers

Perhaps an even more mind boggling and certainly astonishing concept concerning Euler's Identity is that $1^\pi$ can assume infinitely many complex values (Nahin, pg. 166). It can be shown from the formula itself that $1^\pi = \cos(2\pi^2 n) + i\sin(2\pi^2 n)$, for $n = 0, \pm1, \pm2, \cdots$ This is derived by the equation through a direct manipulation. Except for when $n$ does not equal zero, $1^\pi$ will always have an imaginary part. This is because $\sin(2\pi^2 n)$ is never zero save when $n$ is. This seemingly irrational idea, the number one being raised to irrational powers and having complex solutions, clearly demonstrates the depth of Euler's discovery and magnifies the bizarre nature of the interlinking of the complex and real branches of mathematics.

# 3 Modern Applications of Euler's Identity

Euler's formula has much more impact than that of theoretical mathematics. Amazingly, it shows itself as a solution to practical everyday problems. Very much balanced in this respect, between the ideas and the implementation thereof, Euler took seriously the motto of the Berlin Academy of Sciences: "*Theoria cum praxi*," or, "Theory with practice" (Nahin [4], pg. 256).

Euler's formula abounds in electronics and engineering. Underlying electric functions and laws of capacitance and reactance is the famous identity. It is implemented in linear, time-invariant function input-output machines, otherwise known as LTI boxes. These "boxes" take values of $x_n(t)$ and turns them into corresponding values of $Y_n(t)$ through the multiplication of each $x_n(t)$ by some constant $c$, which could be a complex number. The particular use of Euler's formula in electrical engineering is in the case of:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

The use of trigonometric functions and $e$ raised to powers involving imaginary numbers is a form of Euler's Identity that engineers use regularly. The context of this equation is bandwidthsmainly in radio station wavelengths. Ideally, engineers could put a radio signal through an LTI box and get a "zero phase distortion" (Nahin, pg. 261), meaning that a signal is perfectly unaltered as it passes through the filter. And for obvious reasons, the less a sound wave is distorted as it is broadcast, the better.

Another real-life example of Euler's equation being applied is in Newton's gravitational law. A relatively foreign addition to Newton's gravitational force equation is

$$F = -G\left(\frac{Mm}{r^2}\right)e^{i\theta}.$$

Because of the advent of Euler's formula, scientists and physicists have the capability to understand and manipulate such an equation involving imaginary numbers. Another astronomical phenomenon that is explained best with the use of complex exponents is planetary retrograde motion. In this example, let the sun be centered on the origin of a coordinate plane, so that the position vector of the earth is $Z_{SE} = e^{2\imath i t}$. Other planets have a similar position vector. At complex values of $t$, the other planets will appear in comparison to earth to reverse direction in their orbits. Euler's formula allows for the understanding of this property and before its advent, complex values of $t$ would have been nonsensical but he made that connection with this variation of his identity: $e^{i\pi} = -1$.

# 4 Conclusion

Euler's contributions to science made ship building, cannon ballistics, fluid dynamics, lunar orbit theory, and other mechanics easier and more accurate to calculate and understand. Even in the field of theology, which was Euler's original destination of study, he made a valuable contribution. While in the court of Russian empress Catherine the Great, Leonhard Euler met a philosopher named Denis Diderot. Euler was a Christian all his life but Diderot was an atheist; Euler presented a mathematical proof of God's existence. According to De Morgan, Euler told him, "Sir, $\frac{a^n+b^n}{z^n} = x$, hence God exists; reply!" Knowing nothing of algebra, or math in any sense, Diderot was humiliated (Brown [6], pg. 2).

Euler's formula is certainly one of the most celebrated and far-reaching discoveries in mathematics. It is deemed the most beautiful equation in math and brings with it a fundamental shift in the exploration of the imaginary field and its relation to the real world. It takes what was once hidden to the forefront and enables even the less mathematical minded to get a glimpse of the unreal and the infinite. Although the formula is known for its theoretical values and insights, the genius of Euler flourished and he took it to new heights with physical constructions which have radically changed the development of technology. From its impact on nearly every branch of theoretical mathematics to real world problem solving, Euler seems to have constructed a joining identity that comes closer to a unified theory in mathematics than Einstein aspired towards in physics.

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